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COMPLETE EXISTENTIAL THEORY OF BERNSTEIN'S SET OF FOUR POSTULATES FOR BOOLEAN ALGEBRAS.*

BY J. S. TAYLOR.

A set of m postulates is considered independent in the ordinary sense if no one of the m postulates is implied by the others. Professor E. H. Moore,† however, has suggested the problem, in connection with sets of postulates, of determining not only the implicational relations existing among the postulates as they stand but also all the implicational relations which exist among properties defined either by the postulates themselves or by the negatives of the postulates. A set of postulates is then said to be *completely independent* if, and only if, no such implicational relations exist. For example, Professor L. L. Dines‡ has shown that while Sheffer's five postulates for Boolean algebras are independent in the ordinary sense, they are not *completely independent* in the sense defined by Professor Moore, since the negative of the first postulate implies the third, fourth, and fifth.

Bernstein's§ set of four postulates for the same discipline is the most concise set yet formulated for Boolean algebras. The determination of all implicational relations existing among these postulates and their negatives is therefore of corresponding interest and simplicity. The four postulates in question define properties of a system Σ in which the primitive ideas are the class K and the binary rule of combination $|$ between elements of K . Any system $\Sigma(K, |)$ of the prescribed type has with respect to the four postulates one of the $2^4 = 16$ characters:

$$(1) \quad (++++), (+++-), \dots (+---), (----);$$

the i th sign of the character being plus or minus according as Σ has or has not the i th property. The body of sixteen propositions stating for the various characters represented in (1) that there exists or does not exist a system having the character in question constitutes what Professor Moore has called "the complete existential theory" of the four postulates.

* Read before the San Francisco Section of the American Mathematical Society, November 25, 1916.

† E. H. Moore, "Introduction to a Form of General Analysis," New Haven Mathematical Colloquium, Yale University Press, page 82.

‡ L. L. Dines, "Complete Existential Theory of Sheffer's Postulates for Boolean Algebras," Bulletin, vol. XXI, Number 4, pp. 183-188.

§ B. A. Bernstein, "A Set of Four Independent Postulates for Boolean Algebras," Transactions, Vol. 17 (1916), pp. 50-52.

1. **Bernstein's postulates. Their complete existential theory.** Bernstein's postulates concerning systems $\Sigma(K, |)$ are:

P_1 . K contains at least two distinct elements.

P_2 . If a, b are elements of K , $a|b$ is an element of K .

Definition 1. $a' = a|a$.

P_3 . If a, b , and the combinations indicated are elements of K ,

$$(b|a)|(b'|a) = a.$$

P_4 . If a, b, c , and the combinations indicated are elements of K ,

$$a'|(b'|c) = [(b|a')|(c'|a')]'.$$

For the four postulates as given the complete existential theory consists of six propositions of non-existence and ten propositions of existence. In particular, postulates P_2 , P_3 , and P_4 are completely independent. The six non-existences arise from the fact that the negative of P_1 implies P_3 and P_4 .

Propositions of Non-Existence.

The six propositions of non-existence, as implied above, may be expressed by the single proposition:

$$(2) \quad \Sigma^{-1} \supset \Sigma^{34}.$$

That is: "If a system Σ does not satisfy P_1 , then it does satisfy P_3 and P_4 ."

The truth of this statement is readily perceived from the following considerations. The hypothesis that P_1 is not satisfied necessitates either K^{null} (a class without any elements) or K^{singular} (a class with only one element). But if K contains no elements, P_3 and P_4 are satisfied vacuously. And if K contains only one element, then P_3 and P_4 are satisfied either evidently or vacuously, according as Σ does or does not satisfy P_2 .

Proposition (2) renders impossible the existence of systems with the following six characters:

$$\begin{aligned} &(-++-); \quad (-+-+); \quad (-+-); \\ &(--+-); \quad (---+); \quad (---). \end{aligned}$$

Propositions of Existence.

The ten propositions of existence are established by the exhibition of ten systems having the ten remaining characters; there are two examples for K^{singular} , seven for K^{dual} , and one for K^{triple} . In each case K contains the least number of elements possible.

Examples for K Singular.

Systems having the characters $(-+++)$ and $(--++)$ respectively are the following:

System I₁. Character $(-+++)$; class composed of single element e_1 with $e_1|e_1 = e_1$.

System I₂. Character $(--++)$; class the same as above with $e_1|e_1 \neq e_1$.

Examples for K dual.

In the examples following $e_i|e_j$ will be defined by means of tables. For instance, if K is a class of two elements, e_1 and e_2 , and if $e_1|e_1 = e_1$, $e_1|e_2 = e_1$, $e_2|e_1 = e_2$, and $e_2|e_2 = x$ (x not a member of K), this will be expressed by the table:

	e_1	e_2
e_1	e_1	e_1
e_2	e_2	x

Throughout the discussion x will mean an element not in K .

System II₁.

	e_1	e_2
$(++++)$	e_1	e_1
	e_2	e_1

System II₂.

	e_1	e_2
$(+++-)$	e_1	e_2
	e_2	e_1

System II₃.

	e_1	e_2
$(++-+)$	e_1	e_1
	e_2	e_1

System II₄.

	e_1	e_2
$(++--)$	e_1	e_1
	e_2	e_2

System II₅.

	e_1	e_2
$(+-++)$	e_1	x
	e_2	x

System II₆.

	e_1	e_2
$(+---)$	e_1	x
	e_2	e_2

System II₇.

	e_1	e_2
$(+---)$	e_1	e_1
	e_2	x

Example for K triple.

System III₁.

		e_1	e_2	e_3
$(+ - + -)$	e_1	e_1	e_2	x
	e_2	e_1	e_2	x
	e_3	x	x	x

That postulates $P_2 - P_4$ are completely independent follows from the fact that systems have been exhibited having the $2^3 = 8$ characters $(+\pm\pm\pm)$.

2. Bernstein's postulates made completely independent.

Now, although Bernstein's postulates have thus been proved to be not *completely independent*, they become so if postulate P_1 is replaced by the following proposition:

P_1' . *K contains at least four distinct elements.*

The complete independence of the modified set of postulates is proved by the exhibition of sixteen systems having the sixteen different characters of (1). Whenever possible, systems whose tables have already been given will be used and will be referred to merely as I_1, I_2 , etc.

Examples for K singular.

$(- + + +), I_1; \quad (- - + +), I_2.$

Examples for K dual.

$(- + + -), II_2; \quad (- + - +), II_3; \quad (- + - -), II_4; \quad (- - - +), II_6;$
 $(- - - -), II_7.$

Example for K triple.

$(- - + -), III_1.$

Examples for K quadruple.

System IV₁.

		e_1	e_2	e_3	e_4
$(++++)$	e_1	e_2	e_1	e_4	e_3
	e_2	e_1	e_1	e_1	e_1
	e_3	e_4	e_1	e_4	e_1
	e_4	e_3	e_1	e_1	e_3

System IV₂.

		e_1	e_2	e_3	e_4
$(+++-)$	e_1	e_1	e_2	e_3	e_4
	e_2	e_1	e_2	e_3	e_4
	e_3	e_1	e_2	e_3	e_4
	e_4	e_1	e_2	e_3	e_4

System IV₃.

		e_1	e_2	e_3	e_4
	e_1	e_1	e_1	e_1	e_1
(++-+)	e_2	e_1	e_1	e_1	e_1
	e_3	e_1	e_1	e_1	e_1
	e_4	e_1	e_1	e_1	e_1

System IV₄.

		e_1	e_2	e_3	e_4
	e_1	e_1	e_1	e_1	e_1
(++--)	e_2	e_2	e_2	e_1	e_1
	e_3	e_1	e_1	e_1	e_1
	e_4	e_1	e_1	e_1	e_1

System IV₅.

		e_1	e_2	e_3	e_4
	e_1	x	x	x	x
(+-++)	e_2	x	x	x	x
	e_3	x	x	x	x
	e_4	x	x	x	x

System IV₆.

		e_1	e_2	e_3	e_4
	e_1	x	e_2	e_2	e_2
(+---)	e_2	e_2	e_2	e_2	e_2
	e_3	e_2	e_2	e_2	e_2
	e_4	e_2	e_2	e_2	e_2

System IV₇.

		e_1	e_2	e_3	e_4
	e_1	e_1	e_2	x	x
(+---)	e_2	x	e_1	x	x
	e_3	x	x	x	x
	e_4	x	x	x	x

System IV₈.

		e_1	e_2	e_3	e_4
	e_1	e_1	e_2	x	x
(+-+-)	e_2	e_1	e_2	x	x
	e_3	x	x	x	x
	e_4	x	x	x	x

That Bernstein's set of four postulates cannot be made completely independent by postulating a minimum of less than four elements may readily be proved. The results obtained in § 1 clearly show that the postulates will not become completely independent if a minimum less than three is postulated. There remains, then, the possibility of postulating a minimum of exactly three elements. If the postulates then become completely independent, it must be possible to devise systems having all the sixteen characters of (1). But this is not the case, for it is then impossible for a system of the character $(--+-)$ to exist. This the reader can verify without serious difficulty.*

* The proof of this may be outlined as follows. Suppose P_1' be replaced by P_1'' : K contains at least three distinct elements. Then if a system of the character $(--+-)$ is to exist, there must be a class K , null, singular, or dual, which contradicts P_2 and P_4 while it satisfies P_3 . But it has been proved above that K -null or K -singular implies P_4 . Let us then consider K to consist of two elements, e_1 and e_2 . Since P_2 must be contradicted, at least one of the four combinations $e_i|e_j$ ($i, j = 1, 2$) must be an element x not in K . But (1) if $e_1|e_1 = x$ or $e_2|e_2 = x$, P_4 is satisfied (vacuously or evidently), as can readily be verified; (2) if $e_1|e_2 = x$, it is found, by letting $e_1|e_1$, $e_2|e_2$, and $e_2|e_1$ take on all possible values, that we get systems which contradict P_3 or which satisfy P_4 , but none of the character $(--+-)$; and (3) if $e_2|e_1 = x$, the results are the same as for the case $e_1|e_2 = x$. No system of the character $(--+-)$ exists under the postulation of P_1'' , and the corresponding set of postulates is therefore not completely independent.

It is interesting to note in this connection that the fact that the number of elements in a

The proposed substitution of P_1' for P_1 is, therefore, the simplest alteration which will make the postulates completely independent. As far as the writer knows, Bernstein's set of postulates as thus modified is the only set of postulates for Boolean algebras whose complete independence has been proved.

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March, 1917.

finite class satisfying Boolean algebras is a power of 2 does not help us in settling the question of complete independence. While one can deduce by its use the fact that a class containing at least three elements necessarily contains at least four, so that Bernstein's set of postulates wherein P_1 is replaced by P_1'' is equivalent to that set when P_1 is replaced by P_1' , it does *not* follow from this that if one set is completely independent, the other one is. Equivalence merely means that each set can be deduced from the other. It does not imply any other relationship between the two sets.